An Introduction to Lie Algebras

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1 Motivation

We want to study the symmetries of space. Some objects that live in space have finite symmetry groups, for instance polygons. However, a sphere has an infinite symmtry group. Moreover, this symmetry group has a natural topology given by its action on $\mathbb{R}^n$; i.e. the topology given by the Euclidean metric on $\mathbb{R}^{n^2}$, the space of matrices, and in fact this also gives a smooth structure. Additionally, composition and inverse are smooth maps.

2 Lie Groups

Definition 1. *A Lie group is a smooth manifold G along with smooth maps $m: G \times G \to G$, i: $G \to G$, and $e: 1 \to G$ such that (G, m, i, e) is a group.*

Example 2. Let $\mathrm{M}(n, \mathbb{R})$ be the manifold of $n \times n$ matrices over $\mathbb{R}$ with the smooth structure given by the usual smooth structure on $\mathbb{R}^{n^2}$. Then there are several Lie groups contained in $\mathrm{M}(n, \mathbb{R})$.

1. $\mathrm{GL}(n, \mathbb{R})$, the set of invertible matrices. A matrix A is invertible if and only if $\det A \neq 0$, so this is an open submanifold of $M(n)$. The operations are given by composition, inverse, and the identity matrix.

2. $\mathrm{SL}(n, \mathbb{R})$, the set of matrices with determinant 1. This is a closed submanifold of $\mathrm{GL}(n, \mathbb{R})$ given by the equation $\det A = 1$.

3. $\mathrm{O}(n, \mathbb{R})$, the set of orthogonal matrices. This is also a closed submanifold of $\mathrm{GL}(n, \mathbb{R})$, given by the equation $A\,A^T = I = A^T A$, where I is the $n \times n$ identity matrix. Now, $|\det A| = |\det A^T|$, so as $\det I = 1$, this implies that $\det A = \pm 1$. Another way of characterizing $\mathrm{O}(n, \mathbb{R})$ is that it is the group of linear transformation that preserve the distance metric on $\mathbb{R}^n$.

4. $\mathrm{SO}(n, \mathbb{R})$, the set of orthogonal matrices with determinant 1. This the intersection of $\mathrm{SL}(n, \mathbb{R})$ and $\mathrm{O}(n, \mathbb{R})$, and is the connected component of $\mathrm{O}(n, \mathbb{R})$ that contains the identity. This is best understood as the group of rotations of $\mathbb{R}^n$. For instance, $\mathrm{SO}(2, \mathbb{R})$ is isomorphic to the group of complex numbers of modulus 1 under multiplication.

Example 3. All of the matrices in the previous example can be taken over $\mathbb{C}$. However, when working over $\mathbb{C}$ we take the conjugate transpose in the definition of $\mathrm{O}(n, \mathbb{C})$, and we call it $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ (unitary group and special unitary group respectively).

Example 4. In both of the previous examples, it is illustrative to take $n = 1$. For instance $\mathrm{GL}(1, \mathbb{R}) = \mathbb{R} \setminus \{0\}$ with the normal multiplicative structure. $\mathrm{U}(1)$ is the set of complex numbers with absolute value 1, which we have noted is the same as $\mathrm{SO}(2, \mathbb{R})$. $\mathrm{SO}(1, \mathbb{R})$ is just the trivial group.

These are all very important Lie groups. In fact, for most intents and purposes, these are the only Lie groups that matter. Ever so often, however, we come across some Lie group that is not given as one of these Lie groups. So what do we do? We try to understand it by looking how it maps into these groups. Additionally, we often understand these Lie groups themselves by looking at how they map into other Lie groups in these examples. To do this, however, we need a definition of Lie group homomorphism.

Definition 5. *Let X and Y be Lie groups. Then a Lie group homomorphism between them is a smooth map $X \overset{f}{\to} Y$ that is also a group homomorphism.*

Example 6. The homomorphism between U(1) and SO(2, ℝ) that we have talked about before is given by

$$\mathrm{e}^{i\theta} \mapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Example 7. For every n, there is a homomorphism from U(1) to U(1) given by $z \mapsto z^n$.

Example 8. For every n, there is a homomorphism from $\mathrm{GL}(n,\, \mathbb{C})$ into $\mathrm{GL}(2n,\, \mathbb{R})$ given by treating an element of $\mathbb{C}^n$ as an element of $\mathbb{R}^{2n}$.

Example 9. Let $A \in \mathrm{GL}(n, \mathbb{R})$. Then the map that sends B to $A^{-1} B A$ is a homomorphism.

Example 10. Note that ℝ with addition is in fact a Lie algebra. Henceforth, when we talk about ℝ, we will mean ℝ with its additive structure. There is a Lie algebra homomorphism from $\mathbb{R} \to \mathrm{GL}(1, \mathbb{R})$ given by $x \mapsto \mathrm{e}^x$. There is also a Lie algebra homomorphism from ℝ to U(1) given by $x \mapsto \mathrm{e}^{ix}$. This "winds around the circle".

Example 10 is an important example because it can be generalized. Recall that we can define the exponent of a matrix X by

$$\mathrm{e}^X = \sum_{n=0}^{\infty} \frac{X^n}{n!}$$

Moreover, we can prove the normal laws of exponentiation, i.e. $\mathrm{e}^X\, \mathrm{e}^Y = \mathrm{e}^{X+Y}$. Specifically, this means that e^X is always invertible. Additionally, for any matrix $X \in \mathrm{M}(n,\, \mathbb{R})$, we have a homomorphism $\mathbb{R} \to \mathrm{GL}(n)$ given by $t \mapsto \mathrm{e}^{tX}$. These sort of maps may be familiar from the study of quantum mechanics. Specifically, this sort of map comes up in solutions to the Schrödinger equation

$$i\,\hbar \frac{\mathrm{d}\psi}{\mathrm{d}t} = H\,\psi$$

where H is a self-adjoint linear operator on a Hilbert space that ψ is a member of. The solutions to this equation are of the form

$$t \mapsto \mathrm{e}^{-itH/\hbar}\,\psi_0$$

for some ψ_0. Note that ψ_0 is arbitrary, really the important part is the map

$$t \mapsto \mathrm{e}^{-itH/\hbar}$$

One important property of this map is that $\mathrm{e}^{-itH/\hbar}$ is actually unitary. This is because

$$\left(\mathrm{e}^{-itH/\hbar}\right)^\dagger = \mathrm{e}^{itH^\dagger/\hbar} = \mathrm{e}^{itH/\hbar} = \left(\mathrm{e}^{-itH/\hbar}\right)^{-1}$$

where we have used self-adjointness in the second step to derive $H^\dagger = H$. Therefore, if the Hilbert space is $\mathbb{C}^n$, this gives a map from ℝ to $\mathrm{U}(n)$.

Now, self-adjoint matrices are much easier to work with than unitary matrices. The only things that you can do with unitary matrices that keep them unitary are composing them and taking inverses. However, self-adjoint matrices form a vector space, so we can talk about bases and linear combinations. We have this connection between self-adjoint matrices and unitary matrices given by the above equation, so it would be nice if we could work with self-adjoint matrices and use them to derive properties of unitary matrices. To do this, we introduce the concept of a Lie algebra.

3 Lie Algebras

We saw earlier that there are some special maps $\mathbb{R} \to \mathrm{U}(n)$ given by $t \mapsto \mathrm{e}^{tA}$ where $A = -A^\dagger$ is a skew self adjoint matrix. Suppose φ is such a map for the matrix A. Then note that

$$\varphi'(0) = A\,\mathrm{e}^{0A} = A\,I = A$$

Therefore, the tangent space of $\mathrm{U}(n)$ at the identity matrix has a subspace consiting of all skew self-adjoint matrices. To show that in fact the tangent space of $\mathrm{U}(n)$ at the identity matrix consists solely of this subspace, let $\varphi: (-\epsilon, \epsilon) \to \mathrm{U}(n)$ be a representative for an element of the tangent space. Then

$$\begin{aligned}
\varphi'(0) &= \lim_{h\to 0} \frac{\varphi(h) - \varphi(0)}{h} \\
&= \lim_{h\to 0} \varphi(h)\,\frac{\varphi(0) - \varphi(h)^{-1}}{h} \\
&= \lim_{h\to 0} \varphi(h)\,\lim_{h\to 0} \frac{\varphi(0) - \varphi(h)^\dagger}{h} \\
&= -\varphi(0)\,\lim_{h\to 0} \frac{\varphi^\dagger(h) - \varphi(0)}{h} \\
&= -\varphi'(0)^\dagger
\end{aligned}$$

So it actually turns out that in this Lie group, all of the elements of the tangent space come from maps $\mathbb{R} \to \mathrm{U}(n)$. Moreover, if we require these maps to be Lie group maps, any element v of the tangent space come from a *unique* Lie group map $\varphi_v: \mathbb{R} \to \mathrm{U}(n)$, namely

$$\varphi_v(t) = \mathrm{e}^{t\varphi_v'(0)}$$

As is often the case in math, we take an observation that is true in a certain domain, and then use it for the basis of a more general definition. So in general, we will study a Lie group G by studying the properties of its tangent space at the identity element, $T_e\,G$. There is also a general map from $T_e\,G$ to G given by $v \mapsto \varphi_v(1)$, where $\varphi_v: \mathbb{R} \to G$ is the unique Lie group map such that $\varphi_v'(0) = v$. φ_v exists in general because if $\varphi: (-\epsilon, \epsilon) \to G$ is a representative for v, we can define $\varphi_v(k\,\delta) = \varphi(\delta)^k$ and let δ go to 0. Therefore, there is a close connection between the Lie group and its tangent space at the identity element. However, just looking at this tangent space unfortunately does not give us much information, because vector spaces are kind of boring. To learn more about the Lie group without leaving the land of the tangent space, we put more structure on the tangent space that is derived from the Lie algebra. Before we get into this extra structure, let's give some motivation for why the extra structure is useful

First, we recall a theorem from quantum mechanics. Suppose that H is the Hamiltonian operator for some system, and A is a self-adjoint operator corresponding to an observable we care about. Then if we are in the Heisenberg picture of matrix mechanics, the equation for $A(t)$ is

$$i\,\hbar\,\frac{\mathrm{d}}{\mathrm{d}t}\,A(t) = [H, A(t)]$$

where

$$[H, A(t)] = H\,A(t) - A(t)\,H$$

We call $[A, B]$ the *commutator* of A and B, because it measures how far away A and B are from commuting. Specifically, if $[A, B] = 0$ then $A B = B A$.

Example 11. Consider the Hilbert space for the spin of a spin-$\frac{1}{2}$ particle. There are three famous self-adjoint operators on this space (S_z, S_x, and S_y) that give the spin in the z, x, and y directions respectively. These operators satisfy the commutation relationship

$$[S_i, S_j] = \varepsilon_{ijk}\, i\, \hbar\, S_k$$

where ε_{ijk} is the Levi-Civita symbol that gives the sign of the permutation of z, x, y given by i, j, k. Suppose that there is an external magnetic field in the z direction, so that we have a Hamiltonian given by

$$H = \omega_0\, S_z$$

Then consider an observable A such that $A(0) = S_x$.

$$i\,\hbar \frac{\mathrm{d}}{\mathrm{d}t} A(t) = \omega_0 [S_z, A(t)]$$

We write $A(t) = A_z(t)\, S_z + A_x(t)\, S_x + A_y(t)\, S_y + A_0(t)\, S_0$, where $S_0 = I$ and $A_i(t)$ is real. We can do this because the S_i form a basis over $\mathbb{R}$ for the space of self-adjoint matrices. Now, as S_z and S_0 commute with H, $A_z(t)$ and $A_0(t)$ are conserved quantities. Therefore $A_z(t) = A_z(0) = 0$, and similarly $A_0(t) = 0$. We can use this to rewrite the previous equation

$$\begin{aligned} i\,\hbar \frac{\mathrm{d}}{\mathrm{d}t}(A_x(t)\, S_x + A_y(t)\, S_y) &= \omega_0(A_x(t)\, [S_z, S_x] + A_y(t)\, [S_z, S_y]) \\ &= i\,\hbar\,\omega_0 (A_x(t)\, S_y - A_y(t)\, S_x) \end{aligned}$$

and this can be split into simultaneous equations

$$\begin{aligned} A_x'(t) &= -\omega_0\, A_y(t) \\ A_y'(t) &= \omega_0\, A_x(t) \\ A_x(0) &= 1 \\ A_y(0) &= 0 \end{aligned}$$

with solution $A_x(t) = \cos(\omega_0\, t)$, $A_y(t) = \sin(\omega_0\, t)$.

Using the commutator, we managed to completely avoid working with coordinate representations of the spin matrices.

Now, for a general Lie group, the tangent space at the identity is not in natural bijection with a vector space comprised of matrices, so there is no way to define the commutator as $A\,B - B\,A$. However, we can still define something similar, using the composition operator on the Lie group. Namely, if v_1, v_2 are two tangent vectors with φ_1, φ_2: $(-\epsilon, \epsilon) \rightarrow \mathrm{U}(n)$ representatives of the underlying equivalence classes, then we define $[v_1, v_2]$ to be the equivalence class corresponding to

$$t \mapsto \varphi_1(t)\, \varphi_2(t)\, \varphi_1(t)^{-1}\, \varphi_2(t)^{-1}$$

It turns out that in the case of matrix groups where the tangent space is a vector space of matrices, this is just the normal commutator. However, the advantage of this definition is that it works in general. We call this operation the "Lie bracket".

Definition 12. *A Lie bracket is a binary operation on a vector space that is*

1. *bilinear, i.e.*

   a) $[a\,v, w] = [v, a\,w] = a\,[v, w]$

   b) $[v_1 + v_2, w] = [v_1, w] + [v_2, w]$ *and* $[v, w_1 + w_2] = [v, w_1] + [v, w_2]$

2. *anticommutative, i.e.* $[v, w] = -[w, v]$

3. *and satisfies the Jacobi identity*

$$[u, [v, w]] + [w, [u, v]] + [v, [w, u]] = 0$$

Lemma 13. *The Lie bracket associated to the tangent space to a Lie group at a point is in fact a Lie bracket.*

Proof. *Omitted.* □

Definition 14. *A Lie algebra is a vector space equipped with a Lie bracket. A Lie algebra homomorphism is a linear map that preserves the Lie bracket. The tangent space to a Lie group G equipped with the Lie bracket described above is called the associated Lie algebra to G, and normally denoted by $\mathfrak{g}$ (Lie groups are denoted by capital letters, Lie algebra are denoted by lowercase fraktur letters).*

Example 15. There are Lie algebras associated to all of the matrix algebras given before, all of which have the commutator for their Lie bracket

1. $\mathfrak{gl}(n, \mathbb{R})$ is the Lie algebra of all $n \times n$ matrices

2. $\mathfrak{sl}(n, \mathbb{R})$ is the Lie algebra of $n \times n$ matrices with trace 0, because $\operatorname{tr}(A) = \log(\det(\mathrm{e}^A)) = \log(1) = 0$.

3. As $\mathrm{O}(n, \mathbb{R})$ has two components, and the one containing the identity is precisely $\mathrm{SO}(n, \mathbb{R})$, so $\mathfrak{o}(n, R) = \mathfrak{so}(n, \mathbb{R})$. Just as we showed before with complex matrices, $\mathfrak{so}(n, \mathbb{R})$ is the Lie algebra of skew self-adjoint matrices.

The natural question to ask at this point is whether the construction of the associated Lie algebra is a *functorial* construction. That is, is it the action on objects of some functor from the category of Lie groups and Lie group homomorphisms to the category of Lie algebras and Lie algebra homomorphisms. Fortunately, this is the case, and it can be shown fairly easily. If G and L are two Lie groups, $f: G \to L$ is a Lie group homomorphism, f^* is already a linear map from $\mathfrak{g}$ to $\mathfrak{l}$. It remains to show that if $v_1 = [\varphi_1], v_2 = [\varphi_2] \in \mathfrak{g}$, then $f^*([v_1, v_2]) = [f^*\,v_1, f^*\,v_2]$. This follows from the fact that for all t,

$$f(\varphi_1(t)\,\varphi_2(t)\,\varphi_1(t)^{-1}\,\varphi_2(t)^{-1}) = f(\varphi_1(t))\,f(\varphi_2(t))\,f(\varphi_1(t))^{-1}\,f\,(\varphi_2(t))^{-1}$$

The natural next question is *how much* can this construction tell us about Lie groups. Specifically, if two Lie groups have isomorphic Lie algebras, does follow that the Lie groups are isomorphic? However, we have already seen that this is not the case, because $\mathfrak{so}(n, \mathbb{R}) = \mathfrak{o}(n, \mathbb{R})$, but $\mathrm{SO}(n, \mathbb{R}) \neq \mathrm{O}(n, \mathbb{R})$. This makes sense, because a Lie algebra could only know anything about the connected component that it was in. Perhaps then it could be true that two connected Lie groups are isomorphic if and only if their Lie algebras are equal. Alas, this is also not the case.

Example 16. There is a two-to-one covering of $\mathrm{SO}(3, \mathbb{R})$ (which is homeomorphic to $\mathbb{R}\mathrm{P}^3$) by $\mathrm{SU}(2)$ (which is homememomorphic to S^3) that induces an isomorphism of Lie algebras $\mathfrak{so}(3, \mathbb{R}) \cong \mathfrak{u}(2)$.

In general, it is unreasonable to expect Lie algebras to know anything about topological properties of Lie groups because they are only defined locally. However, given this caveat, Lie algebras are as strong as you could reasonably expect them to be.

Proposition 17. *If G_1 and G_2 are simply connected Lie groups with isomorphic Lie algebras, then G_1 and G_2 are isomorphic.*

I think that this suffices for a basic introduction to Lie algebras, so I'm going to hold off on saying anything else for now.