An Invitation to Geometric Type Theory

Owen Lynch[1,2] David Jaz Myers[1]

[1]Topos Institute

[2]Oxford University

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Outline

Synthetic mathematics

Mathematics	Geometric theory
Synthetic domain theory	Theory of a bounded distributive lattice
Synthetic algebraic geometry	Theory of a commutative ring
Synthetic stone duality	Theory of a boolean algebra
Synthetic higher category theory	Theory of a bounded linear order
Synthetic differential geometry	Theory of an archimedian local C^∞-ring

(Sterling and Ye 2025; Blechschmidt 2017; Cherubini, Coquand, and Hutzler 2024; Cherubini, Coquand, Geerligs, et al. 2024; Riehl and Shulman 2017)

$$\boxed{\text{Geometric theory}} \longrightarrow \boxed{\text{Type theory for its classifying topos}}$$

New horizons for databases

- Let's make a compositional type theory for database schemas, migrations, queries, updates.
- Vickers (1996): *Geometric logic is the logic of statements that admit finite witnesses*.
- Vickers (1992): *Geometric theories as database schema*.
- Such a database should be able to store and query the syntax of type theories.
- Vickers (2017): *Arithmetic universes are a good "primitive recursive" variant of topoi*.
- The Coln Project is currently developing a proof assistant integrated with a database, based on a variant of geometric type theory for arithmetic universes instead of topoi (Coln contributors 2026).

Idea

Goal: A dependent type theory for geometric theories and their models.

Problems:

- Pullbacks in Top are pushouts of logoi, which only exist quite weakly.
- Top is not locally cartesian closed, does not have a universe of topoi.
- We also care about the internal language of individual topoi.

Solution: A 2-level type theory where:

- Contexts are *Elephant-style geometric theories*, substitution is strict.
- There is a universe of *geometric constructs*
- Function types are appropriately restricted.

Elephant theories

Definition

An **elephant theory** is a 2-functor $\mathrm{Top}^{\mathrm{op}} \to \mathrm{CAT}$. A morphism of elephant theories is a 2-natural transformation (Johnstone 2002).

Idea: T sends $\mathcal{E}$ to category of "models of T internal to $\mathcal{E}$."

Example:

```
theory Semigroup := sig
  t : Set
  m : t → t → t
  assoc : (x y z : t) -> m x (m y z) = m (m x y) z
end
```

Non-example: $(X : \mathrm{Set}) \times (X \cong X^X)$.

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Geometric constructs

Definition

Let $\mathsf{Set} \colon \mathsf{Top}^{\mathrm{op}} \to \mathsf{CAT}$ be the functor sending $\mathcal{E}$ to $\mathsf{Sh}(\mathcal{E})$.

Definition

Let T be an elephant theory. A **geometric construct** in T is a morphism $A \colon T \Rightarrow \mathsf{Set}$.

Example:

```
def projection (G : Semigroup) : Set := sig
  x : G.t
  is-proj : G.m x x = x
end
```

Simple extensions

Definition

Let T be an elephant theory.

- A **simple functional extension** of T is another elephant theory T' given as the inserter of two geometric constructs $A, B \colon T \Rightarrow \mathsf{Set}$.
- A **simple equational extension** of T is another elephant theory T' given as the equifier of two morphisms of geometric constructs $\alpha, \beta \colon A \Rrightarrow B$.

```
theory PointedSemigroup := sig
  include Semigroup
  pt : t
end

theory Monoid := sig
  include PointedSemigroup
  lunit : (x : t) -> m pt x = x
  runit : (x : t) -> m x pt = x
end
```

Geometric theories

Definition

A **geometric theory** is an elephant theory T that may be presented by a sequence of theories $T_0, \ldots, T_n = T$, where $T_0 = \mathsf{Set}^n$ and T_{i+1} is a functional extension or equational extension of T_i.

- Equivalently, geometric theories are the smallest class of elephant theories including the object classifier and closed under PIE limits.
- All of the elephant theories that we have discussed are in fact geometric.

Theorem

Every geometric theory T is classified by a topos $\mathcal{S}[T]$, in the sense that $T \cong \mathsf{Top}(-, \mathcal{S}[T])$.

Note: the classifying topos for Set is $\mathcal{S}^{\mathsf{FinSet}}$, not $\mathcal{S}$.

Elephant theories in other doctrines

We can play the same game, replacing topoi with

- Lex (finite limit) categories
- Regular categories
- Lextensive categories
- 2-algebras for any sub-2-monad of the infinite colimit completion monad on Lex (subgoemetric logics, Liberti and Ye 2026)
- Arithmetic universes

All that changes is replacing "geometric construct" with "lex construct", "regular construct", "lextensive construct", "arithmetic construct".

Overview

We simultaneously build a type theory and a model.

1. Basic judgment structure
2. A universe
3. What standard type formers does our setup support?

Analogous to the "theory of signatures" which formalizes generalized algebraic theories (Kovács 2022).

Basic judgments for a category with families

Judgment	Semantics in the **geometric model**
Γ ctx	$\Gamma \colon \mathsf{Top}^{\mathrm{op}} \longrightarrow \mathsf{CAT}$ is an geometric theory
$\Gamma \vdash \delta \colon \Delta$	$\delta \colon \Gamma \Rightarrow \Delta$ is a 2-natural transformation
$\Gamma \vdash A$ type	$A \colon (\int \Gamma)^{\mathrm{op}} \longrightarrow \mathsf{CAT}$ is an geometric theory relative to Γ
$\Gamma \vdash a : A$	$a(\mathcal{E} \colon \mathsf{Top}, M \colon \Gamma(\mathcal{E})) \colon A(\mathcal{E}, M)$ is a natural section of A

The context extension $\Gamma \triangleright A$ is interpreted by

$$(\Gamma \triangleright A)(\mathcal{E}) = (M \colon \Gamma(\mathcal{E})) \times A(\mathcal{E}, M)$$

Note: $\int \Gamma \cong \mathsf{Top}/\mathcal{S}[\Gamma]$. Moreover, $\mathcal{S}[\Gamma \triangleright A]$, is an over-topos of $\mathcal{S}[\Gamma]$, and a induces a section of $\mathcal{S}[\Gamma \triangleright A] \longrightarrow \mathcal{S}[\Gamma]$.

A universe of geometric constructs

First approach:

$$\frac{}{\vdash \mathsf{Set}\ \text{type}} \qquad \frac{\Gamma \vdash c : \mathsf{Set}}{\Gamma \vdash {\uparrow} c\ \text{type}}$$

$\uparrow c$ is "the theory of a global section of c."

$$c : (\mathcal{E} : \mathsf{Top}) \times \Gamma(\mathcal{E}) \to \mathsf{Sh}(\mathcal{E})$$
$$(\uparrow c) : (\mathcal{E} : \mathsf{Top}) \times \Gamma(\mathcal{E}) \to \mathsf{Cat}$$
$$(\uparrow c)(\mathcal{E}, M : \Gamma(\mathcal{E})) = \mathcal{E}(1, c(\mathcal{E}, M))$$

Multi-level type theory via fuss-free universes

A universe internalizes types satisfying a certain smallness judgment (Sterling 2025).

$$\textsf{Set-}\textsc{intro}\quad \frac{\Gamma \vdash A \text{ type} \qquad \Gamma \vdash A \text{ small}}{\Gamma \vdash {\downarrow} A : \mathsf{Set}} \qquad \textsf{Set-}\textsc{elim}\quad \frac{\Gamma \vdash c : \mathsf{Set}}{\Gamma \vdash {\uparrow} c \text{ type} \qquad \Gamma \vdash {\uparrow} c \text{ small}}$$

$$\textsf{Set-}\textsc{compute}\quad \frac{\Gamma \vdash A \text{ type} \qquad \Gamma \vdash A \text{ small}}{\Gamma \vdash {\uparrow}{\downarrow} A \equiv A \text{ type}} \qquad \textsf{Set-}\textsc{expand}\quad \frac{\Gamma \vdash c : \mathsf{Set}}{\Gamma \vdash c \equiv {\downarrow}{\uparrow} c : \mathsf{Set}}$$

A small means A that for some c,

$$A(\mathcal{E}, M) \cong \mathcal{E}(1, c(\mathcal{E}, M))$$

Equivalently, $\mathcal{S}[\Gamma \triangleright A]$ is an etale (slice) over-topos of $\mathcal{S}[\Gamma]$.

Propositions

We can do the same thing for propositions

Prop-INTRO
$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma \vdash A \text{ prop}}{\Gamma \vdash {\downarrow}A : \mathsf{Prop}}$$

Prop-ELIM
$$\frac{\Gamma \vdash c : \mathsf{Prop}}{\Gamma \vdash {\uparrow}c \text{ type} \qquad \Gamma \vdash {\uparrow}c \text{ prop}}$$

Prop-COMPUTE
$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma \vdash A \text{ prop}}{\Gamma \vdash {\uparrow}{\downarrow}A \equiv A \text{ type}}$$

Prop-EXPAND
$$\frac{\Gamma \vdash c : \mathsf{Prop}}{\Gamma \vdash c \equiv {\downarrow}{\uparrow}c : \mathsf{Prop}}$$

Semantically,

$$\mathsf{Prop}(\mathcal{E}) = \mathsf{Sub}_{\mathcal{E}}(1)$$

The classifying topos for Prop is the Sierpinski topos $\mathcal{S}^{\rightarrow}$.

Dependent record types

Theorem

The geometric model supports dependent record types, and they preserve smallness.

Idea:

$$A : (\mathcal{E} : \mathsf{Top}) \times \Gamma(\mathcal{E}) \longrightarrow \mathsf{Cat}$$
$$B : (\mathcal{E} : \mathsf{Top}) \times (\Gamma \triangleright A)(\mathcal{E}) \longrightarrow \mathsf{Cat}$$
$$\Sigma\, A\, B : (\mathcal{E} : \mathsf{Top}) \times \Gamma(\mathcal{E}) \longrightarrow \mathsf{Cat}$$
$$(\Sigma\, A\, B)(\mathcal{E}, M) = (a : A(\mathcal{E}, M)) \times B(\mathcal{E}, (M, a))$$

```
theory PointedSet := sig
  t : Set
  pt : t
end
```

Function types with small domain

Theorem

The geometric model supports dependent function types so long as the domain is small.

Idea:

$$
\begin{aligned}
&A : (\mathcal{E} : \mathsf{Top}) \times \Gamma(\mathcal{E}) \longrightarrow \mathsf{Cat} \\
&A(\mathcal{E}, M) \cong \mathcal{E}(1, (\downarrow A)(\mathcal{E}, M)) \\
&B : (\mathcal{E} : \mathsf{Top}) \times (\Gamma \triangleright A)(\mathcal{E}) \longrightarrow \mathsf{Cat} \\
&\Pi\, A\, B : (\mathcal{E} : \mathsf{Top}) \times \Gamma(\mathcal{E}) \longrightarrow \mathsf{Cat} \\
&(\Pi\, A\, B)(\mathcal{E}, M) = B(\mathcal{E}/(\downarrow A), (M, \Delta_{(\downarrow A)}))
\end{aligned}
$$

Makes sense, because etale over-topoi are exponentiable.

```
theory EndoSpan (A : Set) := A -> A -> Set
```

Other positive type formers

All only at the small level

- Coproduct/sum types
- Quotient types
- Inductive types
- Inductive-inductive types
- Quotient inductive-inductive types

Geometric type theory restricted to small types is the (positive fragment of) the internal language of the classifying topos for the context

A modality for inductive types

Inductive types are initial models of theories that don't involve inductive types

```
theory Nat/motive := sig
  t : Set
  zero : t
  succ : t -> t
end

ind def Nat : Nat/motive := init Nat/motive

ind def ZeroTester : Nat/motive := struct
  t := Bool
  zero := true
  succ := _ => false
end

ind def is-zero (n : Nat.t) : Bool := rec n ZeroTester
```

Relative initial models

Problem:

```
theory TransitiveExt (A : Set) (R : A -> A -> Prop) := sig
  S : A -> A -> Prop
  incl : (a b : A) -> R a b -> S a b
  trans : (a b c : A) -> S a b -> S b c -> S a c
end

# good
ind def transitive-closure (A : Set) (R : A -> A -> Prop) :=
  init (TransitiveExt A R)

# bad
ind def free-point (A : Set) := init A
```

Solution: when using `init`, the context is locked, and locked variables cannot appear in the codomain of term constructors.

Negative types

Definition

Let R be a ring, and A an R-algebra. Let $\mathsf{Spec}\,A = \mathsf{Hom}_{R\text{-}\mathsf{Alg}}(A, R)$ and $\mathcal{O}(X) = X \to R$. Then the *synthetic quasicoherence axiom* for R states that if A is finitely presented the natural homomorphism

$$A \to \mathcal{O}(\mathsf{Spec}\,A)$$

is an isomorphism.

This crucially relies on negative types (function types)!

But negative types do not commute with geometric morphisms...

In order to build the generic proof assistant for synthetic mathematics, we need the full type theory (positive and negative types) of each classifying topos.

Summary

- Motivation 1: Synthetic mathematics in classifying topoi
- Motivation 2: Expressive denotational semantics for databases
- Let's make a geometric type theory!
- Read the elephant carefully: a semantic notion of geometric theory
- Basic judgments of geometric type theory
- A universe
- Basic type formers
- Initial models
- Negative types?

Future work: an endless list of tasks to realize the goals of the two motivations.

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