1.

(a) A monomial ideal is prime if and only if it is generated by monomials of the form x_j. For if $x_j x_i$ is in I and neither x_j nor x_i is in I, then I is not prime by definition. To prove the converse, suppose that I is generated by monomials of the form x_j then for $f(x)g(x) \in I$. Then some x_j divides the lowest-degree term (for the $\mathbb{Z}$-grading) of $f(x)\,g(x)$. Therefore, x_j divides the lowest-degree term of $f(x)$ or $g(x)$. If we subtract that lowest degree to get $f'(x)$, it is still the case that $f'(x)\,g(x) \in I$, and we can repeat until either f or g is 0, at which point we have shown that f or g is composed of components that are divisible by x_j for $x_j \in I$. $\square$

(b) A monomial ideal I is radical if and only if it is generated by monomials $x_1^{a_1} \ldots x_n^{a_n}$ such that $(a_1, \ldots, a_n)$ are relatively prime. For if $k \mid a_i$ for all i, then $x_1^{a_1} \ldots x_n^{a_n} = \left(x_1^{a_1/k} \ldots x_n^{a_n/k}\right)^k$, showing that I is not radical. Conversely, suppose that I is generated by monomials of this form, and that $f^n \in I$. Then we can play a similar game with the lowest-degree component of f as we did earlier to show that $f \in I$. $\square$

2. I claim that $f: A \to S^{-1}A$ is an isomorphism if and only if S consists only of units in A. The if is obvious. To see the only if, note that s has no inverse, and $f(s)$ does have an inverse, so the inverse cannot be hit by f. Now, suppose that f is not an isomorphism, and let $s \in S$ not be a unit in A. Then $(s) \neq A$, however $f((s)) = S^{-1}A$. Therefore, contraction is not a bijection. $\square$

3. We may assume that f is injective, as otherwise we can just take $A = \operatorname{im} f$ and the results remain unchanged. Now, if x and y satisfy monic polynomials over A of degree m and n respectively, then by A-M 5.1, $A[x]$ and $A[y]$ are finitely generated over A. Therefore, $A[x, y]$ is finitely generated over A. Now, $A[x\,y] \subseteq A[x, y]$, so by A-M 5.1 again, $x\,y$ is integral over A. $\square$

4. (A-M p. 44 #5) Let $\sqrt{0}$ be the nilpotent ideal. I claim that $(\sqrt{0})_{\mathfrak{p}}$ is the nilpotent ideal for $A_{\mathfrak{p}}$. This is true because if $\frac{a}{s} \in \sqrt{0_{\mathfrak{p}}}$ then $\frac{a^n}{s^n} = \frac{0}{1}$, so $a^n = 0$, whence $a \in \sqrt{0}$. Now, the assumption that there are no nilpotents in any $A_{\mathfrak{p}}$ is equivalent to saying that $(\sqrt{0})_{\mathfrak{p}} = 0_{\mathfrak{p}}$ for all $\mathfrak{p}$. By A-M proposition 3.8, this implies that $\sqrt{0} = 0$, as required. $\square$

5. (A-M p. 67 #2) Let $\mathfrak{p} = \ker f$. $\mathfrak{p}$ is prime because Ω is an integral domain. By 5.10, there exists $\mathfrak{q} \subseteq B$ such that $\mathfrak{q} \cap A = \mathfrak{p}$. Without loss of generality, assume that $\mathfrak{p} = \mathfrak{q} = \{0\}$; otherwise set A to $A/\mathfrak{p}$ and B to $B/\mathfrak{q}$. Then A is a subring of Ω. Therefore, $\operatorname{Frac} A$ is a subfield of Ω, and $\operatorname{Frac} B$ is an algebraic field extension of $\operatorname{Frac} A$. By the theory of field extensions, we can extend $\operatorname{Frac} A \to \Omega$ to $\operatorname{Frac} B \to \Omega$ (we showed this in 2510). We can then restrict this map to B to get our desired map $B \to \Omega$. $\square$